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Local algebraic instability of shear-flows in the Rayleigh equation

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Abstract
Because of its non-Hermitian property, the stability analysis of a shear-flow system is rather complex. While the Kelvin–Helmholtz instability, being a spatially global and temporally exponential mode, can be detected by a standard analysis of eigenvalues, there may exist a variety of different instabilities. Invoking time asymptotic analysis, the existence of spatially localized and temporally algebraic instability in non-monotonic shear-flows with multiple stationary points is shown.

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1. Introduction

The Rayleigh equation constitutes, perhaps, the simplest formulation of the linear stability of shear-flows. Despite its simple mathematical representation, analytical content of this evolution equation is rather rich. This is due to the non-Hermitian nature of the system.

A ‘cornerstone’ result on the subject is the Rayleigh inflection point theorem [1]—a necessary condition for instability is the presence of an inflection point in the velocity profile. An important generalization of this theorem was due to Fjørtoft [2], who restricted the class of possibly unstable flows.

Both these results rely on the assumption that perturbations can be decomposed in terms of normal modes—the solution to the equation of motion can be represented by the superposition of plane propagating waves. However, due to the non-Hermitian nature of the operator governing the dynamics of such a system, the modal ansatz is not the most suitable way to approach the problem. In fact, non-Hermitian operators are not resolvable in terms of orthogonal and complete sets of eigenmodes, making impossible the formulation of an appropriate spectral theorem. Moreover, the coupling induced by non-Hermitian operators renders a decomposition in terms of orthogonal modes ineffective.

The theory of non-Hermitian operators in finite-dimensional vector spaces, the Jordan canonical form of a matrix, suggests mixed evolutions of the type $t^n e^{i\omega t}$, which for real ω

are called secular. Therefore to give an appropriate description of non-Hermitian systems the modal paradigm, in its classical formulation, turns out to be inappropriate.

Case [3] and Dikii [4], aware of the above difficulties, independently obtained solutions of the Rayleigh equation as an initial value problem using a Laplace transform. In their papers for the first time appropriate attention was paid to the continuous component of the spectrum. The two authors sketched proofs, not complete, that the continuous spectrum is stable. This was rigorously proven a few years later by Rosencrans and Sattinger [5] for monotonic velocity profiles. A rigorous treatment discussing the contribution of the continuous spectrum in the presence of non-monotonic profiles has never been given.

Some researchers, however, showed that the continuous spectra of non-Hermitian fluid systems described by equations very close to Rayleigh equation have secular behaviour characterized by algebraic growth. Chimonas [6] showed how, in the context of stratified shear-flows, contrarily to the estimates of Case [7], the equation of motion admits algebraically growing solutions that are not form preserving and that display strong distortion in their evolution. Smith and Rosenbluth [8] showed that, in the context of Rayleigh equation in cylindrical geometry and for profiles with a stationary point, the azimuthal wavenumber $m = 1$ mode grows algebraically as the square root of time. The origin of this instability, associated with the continuous spectrum of the evolution operator, stays in phase coherence.

In the following sections is presented a class of perturbative solutions of Rayleigh equation in Cartesian geometry, which show algebraic growth. The perturbative analysis is carried out for profiles $f(y)$ for which $|f''(y)| \ll |f(y)|$. The expansion parameter ε, therefore, quantifies the smallness of the curvature of the flow with respect to the flow itself. This class of flows is appropriate for investigating algebraic growth in the Rayleigh equation, because Timofeev [9] showed that, for them, exponentially unstable modes are either absent or confined to very small wavenumbers. The present algebraic instability occurs in a particular non-monotonic velocity profile. The fluctuation grows in a narrow region where a 'resonance condition' holds. These properties are in marked contrast with the well-known Kelvin–Helmholtz instability that grows exponentially in time leaving the shape of the modes unchanged. Differently from Smith and Rosenbluth's solution for a cylindrical geometry, the presence of a single stationary point is not sufficient for the instability to develop; at least two such points are required and the order of one of them has to be equal to or higher than two. The growth occurs at any wave number.

The central goal in this paper is to show how the mathematical complexities of non-Hermitian systems can lead to exotic behaviours of the solutions, far richer than the conventional description based on exponential laws for spectrally resolved eigenmodes, even in linear systems of apparently simple form such as the Rayleigh equation. Such complex evolutions encompass transient phenomena (wave conversions, transient amplification) or, as shown in this paper, algebraic growths and the creation of localized structures. The growth is due to the interplay of criticality and resonance. Namely, the presence of a critical point (a stationary point of order two or higher) in the flow triggers a mechanism of energy transfer to a point where the perturbation vorticity oscillates in resonance with the value of the flow at the critical point. The asymptotic treatment is able to detect such a phenomenon which is overlooked when using traditional methods such as the Laplace transform. This difficulty is connected with the non-monotonicity of the flow which renders inverse Laplace transforming a highly complicated operation. In spite of the peculiarity of the non-monotonic unstable profiles under discussion, the present theory should serve as a caveat in the stability analysis of more complex fluid systems in which the possibilities of resonances, and with them dynamical complexities, are increased. In other words, for non-Hermitian systems exponential stability does not imply stability in a general sense, and the globally uniform behaviour of exponentially evolving modes may hide the formation of singular structures.

2. Formulation

Let us consider an inviscid incompressible fluid. This system is described by Euler equations

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P, \tag{2}$$

where $\mathbf{V}$ and P are the velocity and pressure fields respectively. By taking the curl of equation (2) we obtain the vorticity equation

$$\partial_t \boldsymbol{\Omega} - \nabla \times (\mathbf{V} \times \boldsymbol{\Omega}) = 0, \tag{3}$$

where $\boldsymbol{\Omega} = \nabla \times \mathbf{V}$ is the vorticity. Considering two-dimensional flows, we assume a symmetry with respect to the z component, we can express the velocity field in terms of the stream function Φ as

$$\mathbf{V} = \nabla \Phi \times \mathbf{e}_z, \tag{4}$$

where $\mathbf{e}_z$ is a unit vector normal to the plane of the flow. The vorticity is parallel to $\mathbf{e}_z$ and is related to the stream function via the Poisson equation $\Omega = -\Delta\Phi$ (Δ is the Laplacian). Substituting (4) into (3) gives

$$\partial_t \Omega + \mathbf{V} \cdot \nabla \Omega = 0. \tag{5}$$

By decomposing stream function, velocity and vorticity fields as

$$\Phi = \Phi_0 + \phi, \qquad \mathbf{V} = \mathbf{V}_0 + \mathbf{v}, \qquad \Omega = \Omega_0 + \omega, \tag{6}$$

where Φ_0, $\mathbf{V}_0$ and Ω_0 represent the equilibrium and ϕ, $\mathbf{v}$ and ω the perturbation fields, equation (5) can be linearized to give

$$\partial_t \omega + \mathbf{V}_0 \cdot \nabla \omega + \mathbf{v} \cdot \nabla \Omega_0 = 0. \tag{7}$$

Assuming a parallel equilibrium flow of the form $\mathbf{V}_0 = (-f(y), 0, 0)$, where $f(y)$ is a bounded function sufficiently smooth, equation (7) becomes

$$(\partial_t - f(y)\partial_x)\Delta\phi = -f''(y)\partial_x\phi, \tag{8}$$

where $' = \partial_y$. Since the ambient field is homogeneous with respect to x, we can decompose ϕ into Fourier modes proportional to $\mathrm{e}^{\mathrm{i}kx}$. Writing $\partial_x = \mathrm{i}k$ with a good quantum number k (in what follows $k > 0$) the Laplacian becomes $\Delta = \partial_y^2 - k^2$ and equation (8) translates as

$$[\partial_t - \mathrm{i}kf(y)]\Delta\phi = -\mathrm{i}kf''(y)\phi, \tag{9}$$

which is the celebrated Rayleigh equation with boundary conditions $\phi(-\infty) = \phi(+\infty) = 0$.

By inverting the Laplacian operator Δ in an unbounded domain, with ϕ vanishing at infinity, we can express the stream function in terms of the vorticity with

$$\phi = -\Delta^{-1}\omega = \frac{1}{2k}\int_{-\infty}^{+\infty} \mathrm{e}^{-k|\bar{y}-y|}\omega(\bar{y}, t)\,\mathrm{d}\bar{y}. \tag{10}$$

3. Perturbative analysis of Rayleigh equation

We now develop a perturbation analysis of the Rayleigh equation when the flow curvature $f''(y)$ is small; for such profiles Timofeev [9] showed that exponential instability is either absent or confined to very small wavenumbers. Assuming $f''(y) = \varepsilon g(y)$, where $g(y)$ is an $O(1)$ function and ε is a small positive parameter quantifying the smallness of the curvature with respect to the flow $f(y)$, equation (9) may be written as

$$[\partial_t - \mathrm{i}kf(y)]\Delta\phi = -\varepsilon \mathrm{i}kg(y)\phi. \tag{11}$$

By substituting (10) into (11) we have

$$[\partial_t - \mathrm{i}kf(y)]\omega = \varepsilon \mathrm{i}\frac{g(y)}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-k|\bar{y}-y|}\omega(\bar{y}, t)\,\mathrm{d}\bar{y}. \tag{12}$$

We look for an approximate solution of equation (12) by means of a perturbative series of the type:

$$\omega = \sum_{m=0}^{+\infty} \varepsilon^m \omega_m. \tag{13}$$

Plugging (13) into (12), we obtain the following recursion equations:

$$[\partial_t - \mathrm{i}kf(y)]\omega_0 = 0, \tag{14}$$

$$[\partial_t - \mathrm{i}kf(y)]\omega_1 = \mathrm{i}\frac{g(y)}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-k|\bar{y}-y|}\omega_0(\bar{y}, t)\,\mathrm{d}\bar{y}, \tag{15}$$

$$\vdots$$

$$[\partial_t - \mathrm{i}kf(y)]\omega_m = \mathrm{i}\frac{g(y)}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-k|\bar{y}-y|}\omega_{m-1}(\bar{y}, t)\,\mathrm{d}\bar{y}. \tag{16}$$

The solution of (14) is $\omega_0 = \Omega_0(y)\mathrm{e}^{\mathrm{i}kf(y)t}$, where $\Omega_0(y)$ is an arbitrary function to be specified by the initial conditions. Substituting it into equation (15) leads to

$$[\partial_t - \mathrm{i}kf(y)]\omega_1 = \mathrm{i}\frac{g(y)}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-k|\bar{y}-y|}\Omega_0(\bar{y})\,\mathrm{e}^{\mathrm{i}kf(\bar{y})t}\,\mathrm{d}\bar{y}. \tag{17}$$

To determine the temporal behaviour of ω_1, we must evaluate the integral on the right-hand side of equation (17):

$$I(y, t) = \int_{-\infty}^{+\infty} \mathrm{e}^{-k|\bar{y}-y|}\Omega_0(\bar{y})\,\mathrm{e}^{\mathrm{i}kf(\bar{y})t}\,\mathrm{d}\bar{y}. \tag{18}$$

We consider now a profile with one stationary point y_b of order $p-1$, which means

$$f'(y_b) = 0, \ldots, f^{(p-1)}(y_b) = 0,\; f^{(p)}(y_b) \neq 0.$$

The calculations relative to an $f(\bar{y})$ having an arbitrary number N of stationary points will be just the repetition N times of the single stationary point case.

For large t the intervals where no stationary points are present give contributions $O(1/t)$ to the integral on the right-hand side of equation (18).

The dominant contribution comes from the neighbourhood of y_b. Equation (18) can be rewritten [10] as

$$I = \mathrm{e}^{-k|y_b-y|}\Omega_0(y_b)\int_{y_b-\delta}^{y_b+\delta} \mathrm{e}^{\mathrm{i}k[f(y_b)+\frac{f^{(p)}(y_b)}{p!}(\bar{y}-y_b)^p]t}\,\mathrm{d}\bar{y} + O(1/t), \tag{19}$$

where δ is a small positive number. Following a standard derivation of the method of stationary phase, we replace δ by ∞. This introduces error terms which vanish like $1/t$ for $t\to\infty$. We then let $w = \bar{y} - y_b$ obtaining

$$I = \mathrm{e}^{-k|y_b-y|}\Omega_0(y_b)\,\mathrm{e}^{\mathrm{i}kf(y_b)t}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}k\frac{f^{(p)}(y_b)}{p!}w^p t}\,\mathrm{d}w + O(1/t). \tag{20}$$

The integral on the right-hand side of equation (20) can be easily computed [10] by means of contour integration in the complex w-plane, giving

$$I = \mathrm{e}^{-k|y_b-y|}\Omega_0(y_b)\,\mathrm{e}^{\mathrm{i}kf(y_b)t}F(\mathrm{e}^{\pm\mathrm{i}\pi/2p})\left(\frac{p!}{kt|f^{(p)}(y_b)|}\right)^{1/p}\frac{\Gamma(1/p)}{p} + O(1/t), \tag{21}$$

where

$$F(\mathrm{e}^{\pm\mathrm{i}\pi/2p}) = \begin{cases} 2\,\mathrm{e}^{+\mathrm{i}\pi/2p} & \text{for} \quad p \text{ even and } f^{(p)}(y_b) > 0,\\ 2\,\mathrm{e}^{-\mathrm{i}\pi/2p} & \text{for} \quad p \text{ even and } f^{(p)}(y_b) < 0,\\ 2\,\cos(\pi/2p) & \text{for} \quad p \text{ odd.}\end{cases} \tag{22}$$

Substituting (21) back into (17) and neglecting the $O(1/t)$ term, we have

$$[\partial_t - \mathrm{i}kf(y)]\omega_1 = \frac{\mathrm{i}}{2}g(y)H(y)A\,\mathrm{e}^{\mathrm{i}kf(y_b)t}t^{-1/p}, \tag{23}$$

where we introduced

$$H(y) = \mathrm{e}^{-k|y_b-y|}\Omega_0(y_b) \qquad \text{and} \qquad A = F(\mathrm{e}^{\pm\mathrm{i}\pi/2p})\left(\frac{p!}{k|f^{(p)}(y_b)|}\right)^{1/p}\frac{\Gamma(1/p)}{p} \tag{24}$$

in order to simplify the notation in the following.

We now turn our attention to the second-order term ω_2. The equation determining it is (16) for $m = 2$. This reads

$$\begin{aligned}[\partial_t - \mathrm{i}kf(y)]\omega_2 = \mathrm{i}\frac{g(y)}{2}\int_{-\infty}^{+\infty}&\mathrm{e}^{-k|y_2-y|}\mathrm{i}\frac{g(y_2)}{2}\mathrm{e}^{\mathrm{i}kf(y_2)t}\\ &\times\left[\int_0^t \mathrm{e}^{-\mathrm{i}kf(y_2)s}\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-k|y_1-y_2|}\Omega_0(y_1)\,\mathrm{e}^{\mathrm{i}kf(y_1)s}\,\mathrm{d}y_1\right)\mathrm{d}s\right]\mathrm{d}y_2.\end{aligned} \tag{25}$$

The integrand on the right-hand side is absolutely integrable and therefore we interchange the order of integration in s and y_2 obtaining

$$\begin{aligned}[\partial_t - \mathrm{i}kf(y)]\omega_2 = \left(\frac{\mathrm{i}}{2}\right)^2 g(y)\int_0^t&\left[\int_{-\infty}^{+\infty}\mathrm{e}^{-k|y_2-y|}g(y_2)\,\mathrm{e}^{\mathrm{i}kf(y_2)(t-s)}\right.\\ &\left.\times\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-k|y_1-y_2|}\Omega_0(y_1)\,\mathrm{e}^{\mathrm{i}kf(y_1)s}\mathrm{d}y_1\right)\mathrm{d}y_2\right]\mathrm{d}s.\end{aligned} \tag{26}$$

For $t\to\infty$ the dominant contribution to the integral on the right-hand side comes again from the neighbourhood of y_b. Equation (26) can be written (see the appendix) as

$$\begin{aligned}[\partial_t - \mathrm{i}kf(y)]\omega_2 = \left(\frac{\mathrm{i}}{2}\right)^2 &g(y)\,\mathrm{e}^{-k|y_b-y|}\Omega_0(y_b)\,\mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{g^{(p-2)}(y_b)}{(p-2)!}\\ &\times\int_0^t\left[\int_{y_b-\delta}^{y_b+\delta}(y_2-y_b)^{p-2}\,\mathrm{e}^{\mathrm{i}k\frac{f^{(p)}(y_b)(y_2-y_b)^p}{p!}(t-s)}\right.\\ &\left.\times\left(\int_{y_b-\delta}^{y_b+\delta}\mathrm{e}^{\mathrm{i}k\frac{f^{(p)}(y_b)(y_1-y_b)^p}{p!}s}\mathrm{d}y_1\right)\mathrm{d}y_2\right]\mathrm{d}s + o(1).\end{aligned} \tag{27}$$

We replace δ by ∞. This introduces error terms $o(1)$ for $t\to\infty$. We then let $w_2 = y_2 - y_b$ and $w_1 = y_1 - y_b$ obtaining

$$[\partial_t - \mathrm{i}kf(y)]\omega_2 = \left(\frac{\mathrm{i}}{2}\right)^2 g(y)\,\mathrm{e}^{-k|y_b-y|}\Omega_0(y_b)\,\mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{g^{(p-2)}(y_b)}{(p-2)!} \times \int_0^t \left[\int_{-\infty}^{+\infty} {w_2}^{p-2}\,\mathrm{e}^{\mathrm{i}k\frac{f^{(p)}(y_b){w_2}^p}{p!}(t-s)} \left(\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}k\frac{f^{(p)}(y_b){w_1}^p}{p!}s}\,\mathrm{d}w_1\right)\mathrm{d}w_2\right]\mathrm{d}s + o(1). \tag{28}$$

The integrals in w_1 and w_2 on the right-hand side of equation (28) can easily be computed by means of contour integration in the complex w_1 and w_2 planes, giving (see the appendix and [10])

$$[\partial_t - \mathrm{i}kf(y)]\omega_2 = \left(\frac{\mathrm{i}}{2}\right)^2 g(y)H(y)A\,\mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{g^{(p-2)}(y_b)}{(p-2)!} \times K\left(\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}}\right)\left(\frac{p!}{k|f^{(p)}(y_b)|}\right)^{\frac{p-1}{p}}\frac{\Gamma(\frac{p-1}{p})}{p}\int_0^t \frac{\mathrm{d}s}{s^{\frac{1}{p}}(t-s)^{\frac{p-1}{p}}} + o(1), \tag{29}$$

where

$$K\left(\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}}\right) = \begin{cases} 2\,\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}} & \text{for} \quad p \text{ even and } f^{(p)}(y_b) > 0, \\ 2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}} & \text{for} \quad p \text{ even and } f^{(p)}(y_b) < 0, \\ +\,\mathrm{i}2\sin\left(\frac{\pi}{2}\frac{p-1}{p}\right) & \text{for} \quad p \text{ odd and } f^{(p)}(y_b) > 0, \\ -\,\mathrm{i}2\sin\left(\frac{\pi}{2}\frac{p-1}{p}\right) & \text{for} \quad p \text{ odd and } f^{(p)}(y_b) < 0. \end{cases} \tag{30}$$

In a more compact form the above equation (29) can be rewritten as

$$[\partial_t - \mathrm{i}kf(y)]\omega_2 = \left(\frac{\mathrm{i}}{2}\right)^2 g(y)H(y)AB\,\mathrm{e}^{\mathrm{i}kf(y_b)t}\int_0^t \frac{\mathrm{d}s}{s^{\frac{1}{p}}(t-s)^{\frac{p-1}{p}}} + o(1), \tag{31}$$

where we introduced

$$B = \frac{g^{(p-2)}(y_b)}{(p-2)!}K\left(\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}}\right)\left(\frac{p!}{k|f^{(p)}(y_b)|}\right)^{\frac{p-1}{p}}\frac{\Gamma\left(\frac{p-1}{p}\right)}{p} \tag{32}$$

in order to simplify the notation.

We can carry out the integral and neglect the $o(1)$ term on the right-hand side of equation (31) obtaining

$$[\partial_t - \mathrm{i}kf(y)]\omega_2 = \left(\frac{\mathrm{i}}{2}\right)^2 g(y)H(y)AB\,\mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma(1)}, \tag{33}$$

where we used the odd-looking ratio on the right-hand side of equation (33) for reasons which will be clear in the following.

The procedure used above can be employed recursively to obtain higher order terms in a straightforward manner. The next three terms were computed explicitly.

For ω_3 we have

$$[\partial_t - \mathrm{i}kf(y)]\omega_3 = \left(\frac{\mathrm{i}}{2}\right)^3 g(y)H(y)AB^2\,\mathrm{e}^{\mathrm{i}kf(y_b)t} \times \int_0^t \frac{1}{(t-s_2)^{\frac{p-1}{p}}}\left(\int_0^{s_2}\frac{1}{{s_1}^{\frac{1}{p}}(s_2-s_1)^{\frac{p-1}{p}}}\mathrm{d}s_1\right)\mathrm{d}s_2 + o\left(t^{\frac{1}{p}}\right), \tag{34}$$

carrying out the integrals and neglecting the $o(t^{\frac{1}{p}})$ term on the right-hand side of equation (34) we obtain

$$[\partial_t - \mathrm{i}kf(y)]\omega_3 = \left(\frac{\mathrm{i}}{2}\right)^3 g(y)H(y)AB^2\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma(1)}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma(1)}{\Gamma\left(1+\frac{1}{p}\right)}t^{\frac{1}{p}}. \tag{35}$$

For ω_4 we have

$$[\partial_t - \mathrm{i}kf(y)]\omega_4 = \left(\frac{\mathrm{i}}{2}\right)^4 g(y)H(y)AB^3\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\int_0^t \frac{1}{(t-s_3)^{\frac{p-1}{p}}}\left[\int_0^{s_3}\frac{1}{(s_3-s_2)^{\frac{p-1}{p}}}\right.$$
$$\left.\left(\int_0^{s_2}\frac{1}{s_1^{\frac{1}{p}}(s_2-s_1)^{\frac{p-1}{p}}}\mathrm{d}s_1\right)\mathrm{d}s_2\right]\mathrm{d}s_3 + o\left(t^{\frac{2}{p}}\right), \tag{36}$$

carrying out the integrals and neglecting the $o(t^{\frac{2}{p}})$ term on the right-hand side of equation (36) we obtain

$$[\partial_t - \mathrm{i}kf(y)]\omega_4 = \left(\frac{\mathrm{i}}{2}\right)^4 g(y)H(y)AB^3\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma(1)}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma(1)}{\Gamma\left(1+\frac{1}{p}\right)}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{2}{p}\right)}t^{\frac{2}{p}}. \tag{37}$$

For ω_5 we have

$$[\partial_t - \mathrm{i}kf(y)]\omega_5 = \left(\frac{\mathrm{i}}{2}\right)^5 g(y)H(y)AB^4\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\int_0^t \frac{1}{(t-s_4)^{\frac{p-1}{p}}}\left\{\int_0^{s_4}\frac{1}{(s_4-s_3)^{\frac{p-1}{p}}}\right.$$
$$\left.\times\left[\int_0^{s_3}\frac{1}{(s_3-s_2)^{\frac{p-1}{p}}}\left(\int_0^{s_2}\frac{1}{s_1^{\frac{1}{p}}(s_2-s_1)^{\frac{p-1}{p}}}\mathrm{d}s_1\right)\mathrm{d}s_2\right]\mathrm{d}s_3\right\}\mathrm{d}s_4 + o\left(t^{\frac{3}{p}}\right), \tag{38}$$

carrying out the integrals and neglecting the $o(t^{\frac{3}{p}})$ term on the right-hand side of equation (38) we obtain

$$[\partial_t - \mathrm{i}kf(y)]\omega_5 = \left(\frac{\mathrm{i}}{2}\right)^5 g(y)H(y)AB^4\, \mathrm{e}^{\mathrm{i}kf(y_b)t}$$
$$\times\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma(1)}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma(1)}{\Gamma\left(1+\frac{1}{p}\right)}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{2}{p}\right)}\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1+\frac{2}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}t^{\frac{3}{p}}. \tag{39}$$

Higher order ω_m can be calculated from

$$[\partial_t - \mathrm{i}kf(y)]\omega_m = \left(\frac{\mathrm{i}}{2}\right)^m g(y)H(y)AB^{m-1}\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\int_0^t \frac{1}{(t-s_{m-1})^{\frac{p-1}{p}}}\left(\int_0^{s_{m-1}}\frac{1}{(s_{m-1}-s_{m-2})^{\frac{p-1}{p}}}\right.$$
$$\left.\times\left(\ldots\ldots\left(\int_0^{s_2}\frac{1}{s_1^{\frac{1}{p}}(s_2-s_1)^{\frac{p-1}{p}}}\mathrm{d}s_1\right)\ldots\ldots\right)\mathrm{d}s_{m-2}\right)\mathrm{d}s_{m-1} + o\left(t^{\frac{m-2}{p}}\right). \tag{40}$$

Simplifying the right-hand sides of equations (35), (37) and (39) we obtain

$$[\partial_t - \mathrm{i}kf(y)]\omega_3 = \left(\frac{\mathrm{i}}{2}\right)^3 g(y)H(y)AB^2\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{\Gamma^2\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}t^{\frac{1}{p}}, \tag{41}$$

$$[\partial_t - \mathrm{i}kf(y)]\omega_4 = \left(\frac{\mathrm{i}}{2}\right)^4 g(y)H(y)AB^3\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{\Gamma^3\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{2}{p}\right)}t^{\frac{2}{p}}, \tag{42}$$

$$[\partial_t - \mathrm{i}kf(y)]\omega_5 = \left(\frac{\mathrm{i}}{2}\right)^5 g(y)H(y)AB^4\, \mathrm{e}^{\mathrm{i}kf(y_b)t}\frac{\Gamma^4\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}t^{\frac{3}{p}}. \tag{43}$$

Now we can integrate equations (23), (33), (41), (42) and (43) with respect to t obtaining

$$\omega_1(y,t) = \frac{\mathrm{i}}{2} g(y) H(y) A \frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} \mathrm{e}^{\mathrm{i}kf(y)t} \int_0^t \frac{\mathrm{e}^{\mathrm{i}k[f(y_b)-f(y)]s}}{s^{1/p}} \,\mathrm{d}s, \tag{44}$$

$$\omega_2(y,t) = \left(\frac{\mathrm{i}}{2}\right)^2 g(y) H(y) AB \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma(1)} \mathrm{e}^{\mathrm{i}kf(y)t} \int_0^t \mathrm{e}^{\mathrm{i}k[f(y_b)-f(y)]s} \,\mathrm{d}s, \tag{45}$$

$$\omega_3(y,t) = \left(\frac{\mathrm{i}}{2}\right)^3 g(y) H(y) AB^2 \frac{\Gamma^2\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} \mathrm{e}^{\mathrm{i}kf(y)t} \int_0^t \mathrm{e}^{\mathrm{i}k[f(y_b)-f(y)]s} s^{1/p} \,\mathrm{d}s, \tag{46}$$

$$\omega_4(y,t) = \left(\frac{\mathrm{i}}{2}\right)^4 g(y) H(y) AB^3 \frac{\Gamma^3\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{2}{p}\right)} \mathrm{e}^{\mathrm{i}kf(y)t} \int_0^t \mathrm{e}^{\mathrm{i}k[f(y_b)-f(y)]s} s^{2/p} \,\mathrm{d}s, \tag{47}$$

$$\omega_5(y,t) = \left(\frac{\mathrm{i}}{2}\right)^5 g(y) H(y) AB^4 \frac{\Gamma^4\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)} \mathrm{e}^{\mathrm{i}kf(y)t} \int_0^t \mathrm{e}^{\mathrm{i}k[f(y_b)-f(y)]s} s^{3/p} \,\mathrm{d}s. \tag{48}$$

In the following two sections we will discuss the summation of the series on the right-hand side of equation (13) in two different types of points. In section 4 points y_* where the resonant condition

$$f(y_*) = f(y_b) \tag{49}$$

holds, will be considered. Points where the above condition is not met will be dealt with in section 5.

4. Resonant points

In points y_* where condition (49) is satisfied, equations (44)–(48) become

$$\omega_1(y_*,t) = \frac{\mathrm{i}}{2} g(y_*) H(y_*) \,\mathrm{e}^{\mathrm{i}kf(y_b)t} A \frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} \frac{p}{p-1} t^{1-\frac{1}{p}}, \tag{50}$$

$$\omega_2(y_*,t) = \left(\frac{\mathrm{i}}{2}\right)^2 g(y_*) H(y_*) \,\mathrm{e}^{\mathrm{i}kf(y_b)t} AB \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma(1)} t, \tag{51}$$

$$\omega_3(y_*,t) = \left(\frac{\mathrm{i}}{2}\right)^3 g(y_*) H(y_*) \,\mathrm{e}^{\mathrm{i}kf(y_b)t} AB^2 \frac{\Gamma^2\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} \frac{p}{p+1} t^{1+\frac{1}{p}}, \tag{52}$$

$$\omega_4(y_*,t) = \left(\frac{\mathrm{i}}{2}\right)^4 g(y_*) H(y_*) \,\mathrm{e}^{\mathrm{i}kf(y_b)t} AB^3 \frac{\Gamma^3\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{2}{p}\right)} \frac{p}{p+2} t^{1+\frac{2}{p}}, \tag{53}$$

$$\omega_5(y_*,t) = \left(\frac{\mathrm{i}}{2}\right)^5 g(y_*) H(y_*) \,\mathrm{e}^{\mathrm{i}kf(y_b)t} AB^4 \frac{\Gamma^4\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)} \frac{p}{p+3} t^{1+\frac{3}{p}}. \tag{54}$$

The recursive pattern has now clearly emerged and in y_* we can write equation (13) as

$$\omega(y_*,t) = \mathrm{e}^{\mathrm{i}kf(y_b)t} \left(\Omega_0(y_*) + \Gamma\left(\frac{1}{p}\right) \Gamma\left(1-\frac{1}{p}\right) g(y_*) H(y_*) AB \left(\frac{\mathrm{i}}{2}\right)^2 \varepsilon^2 \right. \\ \left. \times \sum_{n=-1}^{+\infty} \frac{\left[\frac{\mathrm{i}}{2}\varepsilon\Gamma\left(\frac{1}{p}\right)B\right]^n t^{1+\frac{n}{p}}}{\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{n}{p}\right)} \right). \tag{55}$$

We will discuss now the summation of the series

$$S_p = \sum_{n=-1}^{+\infty} \frac{c^n t^{1+\frac{n}{p}}}{\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{n}{p}\right)}, \tag{56}$$

where $c = \frac{\mathrm{i}}{2}\varepsilon\Gamma\left(\frac{1}{p}\right)B$, for different orders of the stationary point.

(a) $p = 2$

In this case we have

$$S_2 = \frac{\mathrm{e}^{c^2 t} - 1 + \mathrm{e}^{c^2 t}\mathrm{erf}(c\sqrt{t})}{c^2}, \tag{57}$$

which in the limit $t \to \infty$ becomes

$$S_2 = \frac{2\mathrm{e}^{c^2 t} - 1}{c^2} + O\left(\frac{1}{\sqrt{t}}\right). \tag{58}$$

Substituting (58) into (55) and considering that for $p = 2$ we have $B = g(y_b)A$, gives

$$\omega(y_*, t) = \mathrm{e}^{\mathrm{i}kf(y_b)t}\left(\Omega_0(y_*) + \frac{g(y_*)H(y_*)[2\,\mathrm{e}^{c^2 t} - 1]}{g(y_b)}\right). \tag{59}$$

We note that $\mathrm{e}^{c^2 t}$ is always an oscillatory function, therefore for $p = 2$ in y_* the vorticity has an asymptotically oscillatory behaviour without growth.

(b) $p = 3$

We have

$$S_3 = -\frac{\mathrm{e}^{c^3 t}\Gamma\left(\frac{4}{3}, c^3 t\right)}{c^3\Gamma\left(\frac{4}{3}\right)} + \frac{3\,\mathrm{e}^{c^3 t} - 1}{c^3} + \frac{c^2 t^{\frac{2}{3}} - \mathrm{e}^{c^3 t}\Gamma\left(\frac{5}{3}, c^3 t\right)}{c^3\Gamma\left(\frac{5}{3}\right)}. \tag{60}$$

We note that since c^3 is always negative $\mathrm{e}^{c^3 t}$ decays exponentially with t.

In the limit $t \to \infty$ the dominant contribution comes from the first term on the right-hand side of the above equation and we obtain

$$S_3 = -\frac{t^{\frac{1}{3}}}{c^2\Gamma\left(\frac{4}{3}\right)} + O(1). \tag{61}$$

Substituting (61) into (55) we have

$$\omega(y_*, t) = \mathrm{e}^{\mathrm{i}kf(y_b)t}\left(\Omega_0(y_*) - \frac{\Gamma\left(1-\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1+\frac{1}{3}\right)} g(y_*)H(y_*)\frac{A}{B}t^{\frac{1}{3}} + O(1)\right). \tag{62}$$

Since $g(y_b) = 0$, the algebraic growth does not affect the point y_b, but points $y_* \neq y_b$ where condition (49) holds.

(c) $p = 4$

We have

$$S_4 = -\frac{2t^{\frac{1}{2}}}{c^2\sqrt{\pi}} - \frac{\mathrm{e}^{c^4 t}\Gamma\left(\frac{5}{4}, c^4 t\right)}{c^4\Gamma\left(\frac{5}{4}\right)} + \frac{3\,\mathrm{e}^{c^4 t} - 1 + \mathrm{e}^{c^4 t}\mathrm{erf}(c^2\sqrt{t})}{c^4} + \frac{c^3 t^{\frac{3}{4}} - \mathrm{e}^{c^4 t}\Gamma\left(\frac{7}{4}, c^4 t\right)}{c^4\Gamma\left(\frac{7}{4}\right)}. \tag{63}$$

Since c^4 is imaginary $e^{c^4 t}$ oscillates, therefore for $t \to \infty$ the dominant contribution comes from the first term on the right-hand side of the above equation.

$$S_4 = -\frac{2t^{\frac{1}{2}}}{c^2\sqrt{\pi}} + O\big(t^{\frac{1}{4}}\big). \tag{64}$$

Substituting (64) into (55) we have

$$\omega(y_*, t) = e^{ikf(y_b)t}\left(\Omega_0(y_*) - \frac{2\Gamma\big(1-\frac{1}{4}\big)}{\Gamma\big(\frac{1}{4}\big)\sqrt{\pi}} g(y_*)H(y_*)\frac{A}{B}t^{\frac{1}{2}} + O\big(t^{\frac{1}{4}}\big)\right). \tag{65}$$

(d) Larger p

From the above computations it is now evident that at points $y_* \neq y_b$ where condition (49) holds, the vorticity grows like

$$\omega(y_*, t) \sim e^{ikf(y_b)t} t^{\frac{p-2}{p}} + O\big(t^{\frac{p-3}{p}}\big) + O\big(t^{\frac{p-4}{p}}\big) + \cdots + O\big(t^{\frac{1}{p}}\big). \tag{66}$$

In the next section we will discuss the behaviour of the solution at points where condition (49) is not valid.

5. Non-resonant points

We have now to compute the $\omega_i(y, t)$ given in equations (44)–(48) at points y where condition (49) is not obeyed. Since we are interested in possible growth of the vorticity our attention will focus on terms which grow with time.

From equation (44) it can be seen that $\omega_1(y, t)$ does not grow due to the presence of the oscillations induced by the exponential $e^{ik[f(y_b)-f(y)]s}$ which causes cancellations.

From equation (45) we easily find

$$\omega_2(y, t) = \left(\frac{i}{2}\right)^2 g(y)H(y)AB\frac{\Gamma\big(\frac{1}{p}\big)\Gamma\big(1-\frac{1}{p}\big)}{\Gamma(1)}\frac{e^{ikf(y_b)t} - e^{ikf(y)t}}{ik[f(y_b) - f(y)]}. \tag{67}$$

We will compute now $\omega_3(y, t)$, $\omega_4(y, t)$ and $\omega_5(y, t)$.

Before proceeding we recall that for $t \to \infty$ the dominant contribution to integrals of the type

$$J = \int_0^t e^{i\alpha s} s^\beta \, ds, \tag{68}$$

where $\alpha \neq 0$ and $\beta > 0$, reads

$$J \sim \frac{e^{i\alpha t} t^\beta}{i\alpha}. \tag{69}$$

With the help of (69) and keeping only the dominant contribution in the limit of $t \to \infty$, equations (46)–(48) become

$$\omega_3(y, t) = \left(\frac{i}{2}\right)^3 g(y)H(y)AB^2\frac{\Gamma^2\big(\frac{1}{p}\big)\Gamma\big(1-\frac{1}{p}\big)}{\Gamma\big(1+\frac{1}{p}\big)}\frac{e^{ikf(y_b)t} t^{\frac{1}{p}}}{ik[f(y_b) - f(y)]}, \tag{70}$$

$$\omega_4(y, t) = \left(\frac{i}{2}\right)^4 g(y)H(y)AB^3\frac{\Gamma^3\big(\frac{1}{p}\big)\Gamma\big(1-\frac{1}{p}\big)}{\Gamma\big(1+\frac{2}{p}\big)}\frac{e^{ikf(y_b)t} t^{\frac{2}{p}}}{ik[f(y_b) - f(y)]}, \tag{71}$$

$$\omega_5(y,t) = \left(\frac{\mathrm{i}}{2}\right)^5 g(y)H(y)AB^4 \frac{\Gamma^4\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)} \frac{\mathrm{e}^{\mathrm{i}kf(y_b)t}t^{\frac{3}{p}}}{\mathrm{i}k[f(y_b)-f(y)]}. \tag{72}$$

The recursive pattern has now emerged and we write from equation (13)

$$\omega(y,t) = \omega_0(y,t) + \varepsilon\omega_1(y,t) + \frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)g(y)H(y)AB\left(\frac{\mathrm{i}}{2}\right)^2\varepsilon^2}{\mathrm{i}k[f(y_b)-f(y)]} \times \left(-\mathrm{e}^{\mathrm{i}kf(y)t} + \mathrm{e}^{\mathrm{i}kf(y_b)t}\sum_{n=0}^{+\infty}\frac{\left[\frac{\mathrm{i}}{2}\varepsilon\Gamma\left(\frac{1}{p}\right)B\right]^n t^{\frac{n}{p}}}{\Gamma(1+\frac{n}{p})}\right). \tag{73}$$

We will discuss now the summation of the series

$$\sigma_p = \sum_{n=0}^{+\infty}\frac{c^n t^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)}, \tag{74}$$

where as before $c = \frac{\mathrm{i}}{2}\varepsilon\Gamma\left(\frac{1}{p}\right)B$, for different orders of the stationary point.

(a) $p = 2$

In this case we have

$$\sigma_2 = \mathrm{e}^{c^2 t}[1 + \mathrm{erf}(c\sqrt{t})]. \tag{75}$$

In the limit $t \to \infty$ the above equation becomes

$$\sigma_2 = 2\mathrm{e}^{c^2 t} + O\left(\frac{1}{\sqrt{t}}\right). \tag{76}$$

Since c^2 is imaginary no growth occurs.

(b) $p = 3$

We have

$$\sigma_3 = \mathrm{e}^{c^3 t}\left[3 - \frac{\Gamma\left(\frac{1}{3}, c^3 t\right)}{\Gamma\left(\frac{1}{3}\right)} - \frac{\Gamma\left(\frac{2}{3}, c^3 t\right)}{\Gamma\left(\frac{2}{3}\right)}\right]. \tag{77}$$

In the limit $t \to \infty$ we obtain

$$\sigma_3 = 3\mathrm{e}^{c^3 t} + O(t^{-\frac{1}{3}}). \tag{78}$$

Since c^3 is negative, σ_3 is $O(t^{-\frac{1}{3}})$ and no growth occurs.

(c) $p = 4$

We have

$$\sigma_4 = \mathrm{e}^{c^4 t}[3 + \mathrm{erf}(c^2\sqrt{t})] - \mathrm{e}^{c^4 t}\left[\frac{\Gamma\left(\frac{1}{4}, c^4 t\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{\Gamma\left(\frac{3}{4}, c^4 t\right)}{\Gamma\left(\frac{3}{4}\right)}\right], \tag{79}$$

which in the limit $t \to \infty$ becomes

$$\sigma_4 = 4\mathrm{e}^{c^4 t} + O(t^{-\frac{1}{4}}). \tag{80}$$

Since c^4 is imaginary no growth occurs.

(d) Larger p

In this case in the limit $t \to \infty$ we have

$$\sigma_p = p e^{c^p t} + O\left(t^{-\frac{1}{p}}\right). \tag{81}$$

Noting that for p even c^p is imaginary, while for p odd c^p is always negative, we can conclude that no growth pertains.

As the main result of this section we can state that at points y where condition (49) is not obeyed, the vorticity does not grow.

In this section and the previous ones the local algebraic growth was derived for an unbounded domain. Now it will be shown that analogous behaviour pertains to bounded flows. Let us consider the Rayleigh equation in a channel of width $2a$, where a is a real and positive number. The y variable varies in $[-a, a]$. The Green function of the Laplacian operator $\Delta = \partial_y^2 - k^2$ relative to boundary conditions $\phi(a, t) = \phi(-a, t) = 0$ reads

$$G(y, \bar{y}) = -\begin{cases} \dfrac{\sinh[k(y+a)]\sinh[k(\bar{y}-a)]}{k\sinh(2ak)} & (-a \leqslant y \leqslant \bar{y}) \\ \dfrac{\sinh[k(y-a)]\sinh[k(\bar{y}+a)]}{k\sinh(2ak)} & (\bar{y} \leqslant y \leqslant a). \end{cases} \tag{82}$$

Therefore the correspondent of equation (10) is

$$\phi = -\Delta^{-1}\omega = -\int_{-a}^{a} G(y, \bar{y})\omega(\bar{y}, t)\,\mathrm{d}\bar{y}. \tag{83}$$

The same considerations developed in sections 2, 3, 4 and above apply in this case and the same final results hold by simply substituting $G(y, \bar{y})$ in place of the Green function of the Laplacian in an unbounded domain which is $-\frac{e^{-k|\bar{y}-y|}}{2k}$. Therefore all the results previously obtained still hold with $H(y)$ replaced by

$$H_{\text{bounded}}(y) = -2kG(y, y_b)\Omega_0(y_b). \tag{84}$$

Again, at points $y_* \neq y_b$ where the resonant condition given in (49) holds and $p \geqslant 3$, $\omega(y_*, t)$ experiences algebraic growth of the type $e^{ikf(y_b)t} t^{\frac{p-2}{p}}$.

6. Summary

A special class of algebraically growing solutions of the Rayleigh equation has been presented. This instability occurs when the ambient flow is non-monotonic with multiple stationary points. The vorticity perturbation grows, in proportion to $t^{\frac{p-2}{p}}$, locally in the vicinity of the point $y_* \neq y_b$ where the flow velocity $f(y)$ is in resonance with its value at the stationary point y_b (y_b is the point where the critical condition $f'(y_b) = 0$, $f''(y_b) = 0, \ldots, f^{(p-1)}(y_b) = 0$ and $f^{(p)}(y_b) \neq 0$ ($p \geqslant 3$) holds). For this resonant condition to occur at least two stationary points are required. The vorticity perturbation oscillates while growing with a frequency $\omega = kf(y_b)$. This instability, different from the global and exponentially growing Kelvin–Helmholtz modes, does not have a critical wave number.

It is to be noted that Timofeev (see [9], page 205) conjectured on possible unstable oscillations with small increment and with frequency $\omega = kf(y_b)$ in non-monotonic velocity profiles.

In recent years the non-normal dynamics of optimal linear perturbations in shear-flows has been recognized playing a central role in the onset of turbulence in viscous channel flows [11, 12]. It is opportune to compare the localized instability with such perturbations.

The first difference is represented by the dimensionality of the problem. While the local instability is two-dimensional, the mechanism of transient energy growth for optimal perturbations is typically three-dimensional [13] and it is physically due to the combination of vortex-tilting and Reynolds stress mechanisms [14]. Transient energy growth is also possible in two-dimensional optimal perturbations [15] but it is two orders of magnitude lower.

Another important feature to be discussed is represented by viscosity. The solution derived in this paper is inviscid, while optimal perturbations were discovered in viscous fluids. The presence of viscosity changes some mathematical and physical aspects of the problem. In fact the viscous operator admits a discrete and complete set of non-orthogonal eigenmodes [14] without the presence of a continuous spectrum, while in the inviscid case the continuous spectrum is present even in bounded flows [3, 4]. Continuous spectra in non-Hermitian systems can cause pathological behaviours of which the localized growth presented here represents an example.

Moreover, as pointed out by Reddy and Henningson [16], resonance is not necessary for the transient energy growth mechanism of optimal perturbations, while from equation (49) it is clear that resonance is the cause of the local algebraic growth.

However inviscid and viscous dynamics are closely related. In fact, in viscous optimal perturbations the streamwise vortex–streak interaction is inviscid and the perturbation grows as $O(Re)$ (Re is the Reynolds number) at times $t = O(Re)$. The interaction is effective on a time scale $O(Re)$ before viscosity suppresses it. In the case of Couette flow the secular inviscid growth represents a limiting envelope of the transient viscous one in the limit of infinite Re [11]. Therefore as the inviscid absolute algebraic growth of three-dimensional disturbances [17] is modified by the presence of viscous forces in the transient algebraic growth [18], so might happen in the present case.

It is to be noted however that the presence of viscosity restricts the class of possible equilibria. In fact strictly parallel flows obey the stationary viscous equation of motion only if they are linear combinations of the two basic profiles of Poiseuille flow and Couette flow. The class of parallel inviscid equilibrium flows discussed in this paper are not therefore exact viscous equilibria. In discussing their stability in real fluids it is then important to determine whether they can be considered as parallel approximations of nearly parallel flows (we recall that nearly parallel flows are defined as flows of the type $(U(x, y), V(x, y), 0)$ where $V \ll U$ and $\partial_x U \ll \partial_y U$). The investigation of such topics is, however, beyond the scope of this paper and will be a matter for future study.

An important issue to be discussed is represented by the large-time asymptotic character of the solutions derived. In the stationary phase approximation, on which the treatment presented in this paper strongly relies, the time t is large in the sense that the change in the phase $kf(y)t$, in the integral of equation (18), over the integration range is much larger than π and consequently the integrand oscillates many times and cancellations occur. A lower bound for the validity of the treatment is then given by $\tau_{sp} \sim \frac{\pi}{k}$.

Considering that the effects of viscosity will limit the validity of the solution for large times and that their time scale is of order $\tau_v \sim Re$, the analysis should apply to physical perturbations at large Re for intermediate times in the range $\tau_{sp} \ll t \ll \tau_v$.

This is confirmed by the analysis carried out in [19] where the effect of viscosity on the inviscid algebraic instability of Smith and Rosenbluth [8] was studied. It was shown that the early time dynamics is well described by the solution and that the presence of viscous forces inhibits the longtime algebraic growth.

Another limitation to the validity of the solution presented it is connected with its linear character. In fact, the initial algebraic growth may lead to large amplitudes of the perturbations and therefore to nonlinear effects. In this regard it is interesting to consider the results obtained in [19], where the linear algebraic instability of Smith and Rosenbluth was followed in the nonlinear stage. As expected, it was found that at small amplitudes the linear solution represents the evolution of the system. Nonlinear corrections to the Rayleigh equation in cylindrical coordinates lead to secondary instabilities. One can only speculate that similar processes could occur also in the present case.

Due to the specificity of the velocity profiles in which it occurs, the shown localized algebraic growth of perturbations is therefore not an attempt to identify different general mechanisms which may lead to the onset of turbulence in shear-flows, but represents an example of how pathologies inborn in non-Hermitian fluid systems may lead to the creation of localized vortex structures.

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Appendix

It is opportune now to recall some well-known results about the stationary phase method. These can be useful for the analysis presented in the main text.

Let us consider an integral of the type

$$\Upsilon(t) = \int_a^b g(y)\,\mathrm{e}^{\mathrm{i}f(y)t}\,\mathrm{d}y. \tag{A.1}$$

If $f(y)$ has a stationary point of order p at c, $c \in [a, b]$, and $g(c) \neq 0$ then the dominant contribution to $\Upsilon(t)$ in the limit $t \to \infty$ is [10]

$$\Upsilon(t) = \mathrm{e}^{\mathrm{i}f(c)t} F(\mathrm{e}^{\pm\mathrm{i}\pi/2p}) \left(\frac{p!}{t|f^{(p)}(c)|}\right)^{1/p} \frac{\Gamma(1/p)}{p}, \tag{A.2}$$

where

$$F(\mathrm{e}^{\pm\mathrm{i}\pi/2p}) = \begin{cases} 2\mathrm{e}^{+\mathrm{i}\pi/2p} & \text{for} \quad p \text{ even and } f^{(p)}(c) > 0, \\ 2\mathrm{e}^{-\mathrm{i}\pi/2p} & \text{for} \quad p \text{ even and } f^{(p)}(c) < 0, \\ 2\cos(\pi/2p) & \text{for} \quad p \text{ odd}. \end{cases} \tag{A.3}$$

We want to obtain now the asymptotic limit of $\Upsilon(t)$ for $t \to \infty$ when $g(c) = 0$. We consider $g(y) = f^{(2)}(y)$, which is the special case of interest for the paper.

The dominant contribution comes from the neighbourhood of c. We Taylor expand $f(y)$ and $g(y)$ in a neighbourhood of c. Replacing therefore $f(y)$ by $f(c) + \frac{f^{(p)}(c)}{p!}(y-c)^p$ and $g(y)$ by $\frac{f^{(p)}(c)}{(p-2)!}(y-c)^{p-2}$, equation (A.1) can be rewritten as

$$\Upsilon(t) = \int_{c-\delta}^{c+\delta} \frac{f^{(p)}(c)}{(p-2)!}(y-c)^{p-2}\,\mathrm{e}^{\mathrm{i}[f(c)+\frac{f^{(p)}(c)}{p!}(y-c)^p]t}\,\mathrm{d}y + O(1/t), \tag{A.4}$$

where δ is a small positive number. Following a standard derivation of the method of stationary phase [10], δ is replaced by ∞. This introduces error terms which vanish like $1/t$ for $t \to \infty$ and can be neglected. Letting $w = y - c$ gives

$$\Upsilon(t) = \frac{f^{(p)}(c)}{(p-2)!}\mathrm{e}^{\mathrm{i}f(c)t} \int_{-\infty}^{+\infty} w^{p-2}\,\mathrm{e}^{\mathrm{i}\frac{f^{(p)}(c)}{p!}w^p t}\,\mathrm{d}w + O(1/t), \tag{A.5}$$

The above expression changes for p even or odd. More precisely

$$\Upsilon(t) \sim \begin{cases} \dfrac{f^{(p)}(c)}{(p-2)!}\, \mathrm{e}^{\mathrm{i}f(c)t} 2 \displaystyle\int_0^{+\infty} w^{p-2}\, \mathrm{e}^{\mathrm{i}\frac{f^{(p)}(c)}{p!} w^p t} \mathrm{d}w & p \text{ even}, \\ \dfrac{f^{(p)}(c)}{(p-2)!}\, \mathrm{e}^{\mathrm{i}f(c)t} 2\mathrm{i}\, \mathrm{Im} \left(\displaystyle\int_0^{+\infty} w^{p-2}\, \mathrm{e}^{\mathrm{i}\frac{f^{(p)}(c)}{p!} w^p t}\, \mathrm{d}w \right) & p \text{ odd}. \end{cases} \tag{A.6}$$

The integral $\int_0^{+\infty} w^{p-2} \mathrm{e}^{\mathrm{i}\frac{f^{(p)}(c)}{p!} w^p t}\, \mathrm{d}w$ can easily be calculated by means of contour integration in the complex w-plane [10]. The final result for $t \to \infty$ is

$$\Upsilon(t) \sim \frac{f^{(p)}(c)}{(p-2)!} \mathrm{e}^{\mathrm{i}f(c)t} K\left(\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}}\right) \left(\frac{p!}{t|f^{(p)}(c)|}\right)^{\frac{p-1}{p}} \frac{\Gamma\left(\frac{p-1}{p}\right)}{p}, \tag{A.7}$$

where

$$K\left(\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}}\right) = \begin{cases} 2\,\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}} & \text{for} \quad p \text{ even and } f^{(p)}(c) > 0, \\ 2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\frac{p-1}{p}} & \text{for} \quad p \text{ even and } f^{(p)}(c) < 0, \\ \mathrm{i}2 \sin\left(\frac{\pi}{2}\frac{p-1}{p}\right) & \text{for} \quad p \text{ odd and } f^{(p)}(c) > 0, \\ -\mathrm{i}2 \sin\frac{\pi}{2}\left(\frac{p-1}{p}\right) & \text{for} \quad p \text{ odd and } f^{(p)}(c) < 0. \end{cases} \tag{A.8}$$

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